

JOHN ELLIPSOID AND THE CENTER OF MASS OF A CONVEX BODY

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ABSTRACT. It is natural to ask whether the center of mass of a convex body $K \subset \mathbb{R}^n$ lies in its John ellipsoid B_K , i.e., in the maximal volume ellipsoid contained in K . This question is relevant to the efficiency of many algorithms for convex bodies. In this paper, we obtain an unexpected negative result. There exists a convex body $K \subset \mathbb{R}^n$ such that its center of mass does not lie in the John ellipsoid B_K inflated $(1 - C\sqrt{\frac{\log(n)}{n}})n$ times about the center of B_K . Moreover, there exists a polytope $P \subset \mathbb{R}^n$ with $O(n^2)$ facets whose center of mass is not contained in the John ellipsoid B_P inflated $O(\frac{n}{\log(n)})$ times about the center of B_P .

1. INTRODUCTION

Recall that the John ellipsoid B_K of a convex body $K \subset \mathbb{R}^n$ is the maximal volume ellipsoid contained in K . A natural question asked by S. Vempala is whether the center of mass of K lies in a small dilation of its John ellipsoid. The importance of this question stems from its relation to the efficiency of algorithms for convex bodies. The efficiency of many such algorithms depends on the "roundness" of the body. This can be measured in two ways:

- (1) the traditional way, as the ratio of the radii of the circumscribed to the inscribed ball;
- (2) as the ratio of the radii of the smallest ball that contains the most points (say $1/2$ of the volume) to the inscribed ball.

For instance, the complexity of sampling algorithms grows quadratically with the latter ratio. Thus, a common pre-processing step is to find a good rounding—in other words, find an ellipsoid for which this ratio is reasonably small and then map it to the unit ball using an affine transformation. This can be done in a randomized polynomial time algorithm by estimating the inertia ellipsoid (defined by the covariance matrix of a uniform random point from K),

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wherein complexity depends logarithmically on the initial ratio of the radii, but as a large degree polynomial on the dimension. The other possible candidate is the John ellipsoid. This ellipsoid is difficult to construct in general, but for explicit polytopes, a simple iterative algorithm identifies the inscribed ellipsoid of the maximal volume quite efficiently. This algorithm was developed by L. G. Khachiyan [4]. Recently, Y. Lee and A. Sidford have provided a faster algorithm [7]. In contrast to the inertia ellipsoid, whose construction requires sampling, the John ellipsoid is constructed deterministically. The John ellipsoid can be used to reduce the ratio (1) but it can be as large as n , which is the dimension of the body. On the other hand, the inertia ellipsoid yields the bound $O(\sqrt{n})$ for the ratio (2). This raises a question: Does the John ellipsoid also provide a good bound for the ratio (2)? In other words, one can write it as the following conjecture:

Conjecture 1.1. *For any convex body K in \mathbb{R}^n , the John ellipsoid of K scaled by a factor of $O(\sqrt{n})$ about the ellipsoid's center will contain at least half of the volume of K .*

This can be formulated in terms of the center of mass. We will show in Section 4 that Conjecture 1.1 is equivalent to the following conjecture:

Conjecture 1.2. *For any convex body K in \mathbb{R}^n , the John ellipsoid of K scaled by a factor of $O(\sqrt{n})$ about the ellipsoid's center will contain the center of mass of K .*

The main result of this paper is:

Theorem 1.3. *For a sufficiently large $n \in \mathbb{N}$,*

- (1) *There exists a convex body $K \subset \mathbb{R}^n$ such that its center of mass does not lie in the John ellipsoid scaled by a factor of $(1 - C_0 \sqrt{\frac{\log(n)}{n}})n$ about the ellipsoid's center, where $C_0 > 0$ is a universal constant.*
- (2) *There exists a polytope $P \subset \mathbb{R}^n$ with $O(n^2)$ facets such that its center of mass does not lie in the John ellipsoid scaled by a factor of $C_1 \frac{n}{\log(n)}$ about the ellipsoid's center, where $C_1 > 0$ is a universal constant.*

Remark: It is well known that for any convex body $K \subset \mathbb{R}^n$, the John ellipsoid of K scaled by a factor n about the ellipsoid's center contains the original body K . ([5])

Thus, the example in Theorem 1.3(1) is the asymptotically optimal in the sense that $\lim_{n \rightarrow +\infty} \frac{(1 - C_0 \sqrt{\frac{\log(n)}{n}})n}{n} = 1$.

A consequence of this theorem is the following:

Corollary 1.4. *For a sufficiently large $n \in \mathbb{N}$,*

- (1) *There exists a convex body $K \subset \mathbb{R}^n$ such that the center of its John ellipsoid B_K is 0 and*

$$\text{vol}((1 - C'_0 \sqrt{\frac{\log(n)}{n}})nB_K \cap K) \leq \frac{1}{2}\text{vol}(K),$$

where $C'_0 > 0$ is a universal constant.

- (2) *There exists a polytope $P \subset \mathbb{R}^n$ with $O(n^2)$ facets such that the center of its John ellipsoid B_P is 0 and*

$$\text{vol}(C'_1 \frac{n}{\log(n)} B_P \cap P) \leq \frac{1}{2}\text{vol}(P),$$

where $C'_1 > 0$ is a universal constant.

Thus, Conjecture 1.1 and Conjecture 1.2 are not true due to Theorem 1.3 and Corollary 1.4. In particular, both conjectures will not hold even if one restricts the collection of convex bodies to polytopes with $O(n^2)$ facets.

This paper is structured as follows. Section 2 examines the notation and necessary background for the proof of the main theorem. The proof of the main theorem is presented in Section 3. Corollary 1.4 and the relation between Conjecture 1.1 and 1.2 are examined in Section 4.

2. NOTATIONS AND PRELIMIMARIES

Let B_2^n denote the unit Euclidean ball in \mathbb{R}^n and $|\cdot|$ denote the Euclidean norm. Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis for \mathbb{R}^n . For any $x \in \mathbb{R}^n$, let x_i denote its i -th coefficient.

A subset of \mathbb{R}^n is called a convex body if it is a convex, compact set that has a non-empty interior. For a subset $A \subset \mathbb{R}^n$, let 1_A denote the indicator function of A .

For a convex body $K \subset \mathbb{R}^n$, $\text{vol}(K) := \int_{\mathbb{R}^n} 1_K(x) dx$, where the integral is the standard Lebesgue integration on \mathbb{R}^n .

An ellipsoid B_K is the John's ellipsoid of K if $B_K \subset K$; for any other ellipsoid $\mathcal{E} \subset K$, $\text{vol}(B_K) \geq \text{vol}(\mathcal{E})$. It is known that B_K exists and is unique. ([5])

A convex body K is in John's position if the John ellipsoid of K is B_2^n . For any convex body $K \subset \mathbb{R}^n$, there exists an affine transformation T such that TK is in John's position.

If a convex body K contains 0, we can define its radial function $\rho : S^{n-1} \rightarrow \mathbb{R}_+$ by

$$\forall \theta \in S^{n-1}, \rho(\theta) = \max\{t > 0, t\theta \in K\}.$$

The center of mass of a convex body K is defined by

$$x_K := \frac{1}{\text{vol}(K)} \int_K x dx.$$

Let σ_{n-1} denote the normalized Haar measure on S^{n-1} . An absolutely continuous measure μ with density function $\frac{d\mu(x)}{dx} = f(x)$ on \mathbb{R}^n is called log-concave if, for any $\lambda \in (0, 1)$ and $x, y \in \mathbb{R}^n$, we have

$$(1) \quad f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

In this paper, we are interested in a specific class of log-concave measures: For any convex body K , the probability uniformly distributed in K , $\frac{1_K}{\text{vol}(K)} dx$, is a log-concave probability measure. This is due to the fact that indicator functions of convex sets automatically satisfied the inequality (1).

For matrices, let $\text{Tr}(M)$ denote the trace of a square matrix M and I_n denote the identity matrix on \mathbb{R}^n .

Let \mathbb{P} denote the probability and \mathbb{E} denote the expectation. For the standard definition of terms in probability, we refer to E. Çinlar's book. ([2])

2.1. John's Decomposition. Let $K \subset \mathbb{R}^n$ be a convex body in John's position. A point $u \in \mathbb{R}^n$ is a contact point of K and B_2^n if $u \in \partial K \cap \partial B_2^n$. A classical theorem of F. John provides a decomposition of identity in terms of contact points. ([1, p 52])

Theorem 2.1. *Let K be a convex body in \mathbb{R}^n that contains B_2^n . Then, K is in John's position if and only if there exist contact points u_1, \dots, u_m and $c_1, \dots, c_m > 0$ such that*

$$(1) \quad \sum_{i=1}^m c_i u_i \otimes u_i = I_n, \text{ and}$$

$$(2) \quad \sum_{i=1}^m c_i u_i = \vec{0}.$$

Therefore, we can check from its contact points whether a convex body is in John's position.

2.2. Measure Concentration on S^{n-1} . Below we include two measure concentration inequalities on S^{n-1} .

The first inequality is the upper bound for the measure of a spherical cap (see, e.g., [1, p86]):

Proposition 2.2. *Let $A_t = \{\theta \in S^{n-1}, \theta_1 > t\}$, then $\sigma_{n-1}(A_t) \leq 2 \exp(-C_3 t^2 n)$ where $C_3 > 0$ is a universal constant.*

The second inequality is the concentration inequality for Lipschitz continuous functions on the sphere (see, e.g., [8]):

Theorem 2.3 (Measure Concentration on S^{n-1}). *Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant b . Then, for every $t > 0$,*

$$\sigma_{n-1}(\{x \in S^{n-1} : |f(x) - \mathbb{E}(f)| \geq bt\}) \leq 4 \exp(-C_4 t^2 n),$$

where $C_4 > 0$ is a universal constant.

2.3. Measure Concentration for log-concave probability measures. We include Borell's theorem ([1, p. 31]) and its application on comparison of moments ([1, p. 121]):

Theorem 2.4. *Let \mathbb{P} be the probability that is uniformly distributed in a convex body K . Let U be a closed, convex and symmetric set wherein $\mathbb{P}(U) = \delta > 1/2$. Then, for any $t > 1$, we have*

$$\mathbb{P}((tU)^c) \leq \delta \left(\frac{1 - \delta}{\delta} \right)^{\frac{t+1}{2}}.$$

Theorem 2.5. *Let μ be a non-degenerate log-concave probability measure on \mathbb{R}^n . If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a seminorm, then, for any $q > p \geq 1$, we have*

$$(\mathbb{E}|f|^p)^{1/p} \leq (\mathbb{E}|f|^q)^{1/q} \leq C_5 \frac{q}{p} (\mathbb{E}|f|^p)^{1/p},$$

where $C_5 > 0$ is some universal constant.

In the end, we include one more theorem about log-concave probability measures (Corollary 1 in [6]):

Theorem 2.6. *For each $0 < b < 1$ there exists a constant C_b such that for every log-concave probability measure μ and every measurable convex symmetric set U with $\mu(U) = b$ we have*

$$\mu(tU) \leq C_b t \mu(U) \text{ for } t \in [0, 1].$$

3. PROOF OF THE MAIN THEOREM

Since the result of Theorem 1.3 is not affected by applying an affine transformation on K or P , Theorem 1.3 can be rephrased as follows:

Theorem 3.1. *For a sufficiently large $n \in \mathbb{N}$,*

- (1) *There exists a convex body $K \subset \mathbb{R}^n$ in John's position such that $|x_K| \geq (1 - C_0 \sqrt{\frac{\log(n)}{n}})n$ where $C_0 > 0$ is a universal constant.*

- (2) *There exists a convex polytope $P \subset \mathbb{R}^n$ in John's position with $O(n^2)$ facets such that $|x_P| \geq C_1 \frac{n}{\log(n)}$ where $C_1 > 0$ is a universal constant.*

We write points in the form $x = (y, t)$ where $y \in \mathbb{R}^{n-1}$ corresponds to $\{e_i\}_{i=1}^{n-1}$ and $t \in \mathbb{R}$ corresponds to e_n . For a convex body K , we write $x_K := (y_K, t_K)$. Observe that, for $R > 0$,

$$\begin{aligned} t_K - R &\geq 0 \\ \Leftrightarrow \frac{1}{\text{vol}(K)} \int_K (t - R) dx &\geq 0 \\ \Leftrightarrow \int_K (t - R) dx &\geq 0. \end{aligned}$$

Also, for a convex body $K \subset \mathbb{R}^n$, let $K_t := \{y \in \mathbb{R}^{n-1}, (y, t) \in K\}$, which is a slice of the convex body K . Let $[a_K, b_K]$ be the orthogonal projection of K to the span of e_n .

Assuming $0 \in K_t$ for all $t \in [a_K, b_K]$, let $\rho_K(\cdot, t)$ denote the radial function of K_t as a convex body in \mathbb{R}^{n-1} . Then,

$$\begin{aligned} &\int_K (t - R) dx \\ &= \int_{a_K}^{b_K} (t - R) \int_{K_t} dy dt \\ &= \int_{a_K}^{b_K} (t - R) (n-1) \kappa_{n-1} \int_0^{\rho_K(\theta, t)} \int_{S^{n-2}} r^{n-2} dr d\sigma_{n-2}(\theta) dt \\ &= \kappa_{n-1} \int_{S^{n-2}} \int_{a_K}^{b_K} \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta), \end{aligned}$$

where κ_n denote the volume of B_2^n . With $|x_K| \geq |t_K|$, we conclude

$$\begin{aligned} (2) \quad &\int_{S^{n-2}} \int_{a_K}^{b_K} \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \geq 0 \\ &\Rightarrow |x_K| \geq R. \end{aligned}$$

Before moving on to the proof of the main theorem, we examine two simple convex bodies in \mathbb{R}^n .

Let $0 \in B \subset \mathbb{R}^{n-1}$ be a $n-1$ dimensional convex body. We define $B_1, B_2 \subset \mathbb{R}^n$ as

$$\begin{aligned}
B_1 &:= \{(y, t) \in \mathbb{R}^n, y \in B \text{ and } t \in [0, n+1]\}, \text{ and} \\
B_2 &:= \{(y, t) \in \mathbb{R}^n, y \in \frac{t}{n+1}B \text{ and } t \in [0, n+1]\}.
\end{aligned}$$

In other words, B_1 is a cylinder and B_2 is a cone. Both of them have the same base B and height $n+1$. We have $t_{B_1} = \frac{n+1}{2}$.

For t_{B_2} , using the fact that B_2 is a cone, we have

$$\begin{aligned}
t_{B_2} &= \langle x_{B_2}, e_n \rangle \\
&= \frac{1}{\text{vol}(B_2)} \int_{B_2} t dx \\
&= \frac{n}{(n+1)\text{vol}(B)} \int_0^{n+1} \text{vol}(B) \left(\frac{t}{n+1}\right)^{n-1} t dt \\
&= n.
\end{aligned}$$

Comparing these two examples, we see that x_{B_2} is much closer to its base. For the same reason, the convex hull of B_2^n and ne_n , which is in John's position, has a center of mass that lies in B_2^n , because its shape is similar to that of a cone.

We will construct examples in the Theorem 3.1 as the intersection of two convex bodies, $Q \cap L$. Q and L will satisfy the following:

- (1) Q is in John's position. L contains B_2^n . Thus, $Q \cap L$ is also in John's position.
- (2) L will be a cone (or a cylinder) with the property that $Q \cap L$ and L have a similar shape. Therefore, $x_{Q \cap L}$ behaves like the center of mass of a cone (or a cylinder).

3.1. Construction of Q . The following proposition is related to the contact points decomposition of the identity:

Proposition 3.2. *Let u_1, \dots, u_m be unit vectors in $\mathbb{R}^{n-1} = \text{span}\{e_1, \dots, e_{n-1}\} \subset \mathbb{R}^n$, and $c_1, \dots, c_m > 0$ be some positive numbers such that*

$$\sum_{i=1}^m c_i u_i \otimes u_i = I_{n-1} \quad \text{and} \quad \sum_{i=1}^m c_i u_i = \vec{0}.$$

Set $v_i = (\sqrt{1 - \frac{1}{n^2}} u_i, \frac{1}{n}) \in \mathbb{R}^n$ for $i = 1, \dots, m$ and $v_0 = (\vec{0}, -1)$. With $c'_i = \frac{c_i}{1 - \frac{1}{n^2}}$ and $c'_0 = \frac{n}{n+1}$, we obtain

$$\sum_{i=0}^m c'_i v_i \otimes v_i = I_n, \quad \text{and} \quad \sum_{i=0}^m c'_i v_i = \vec{0}.$$

Proof. From the definition of v_i , we have

$$v_i \otimes v_i = \frac{1}{n^2} e_n \otimes e_n + \frac{1}{n} \sqrt{1 - \frac{1}{n^2}} (e_n \otimes u_i + u_i \otimes e_n) + \left(1 - \frac{1}{n^2}\right) u_i \otimes u_i.$$

We know that $n - 1 = \text{Tr}(I_{n-1}) = \text{Tr}(\sum_{i=1}^m c_i u_i \otimes u_i) = \sum_{i=1}^m c_i$.
Thus, we have

$$\sum_{i=0}^m c_i v_i \otimes v_i = \frac{n-1}{n^2} e_n \otimes e_n + \left(1 - \frac{1}{n^2}\right) I_{n-1}.$$

where we use the fact that $\sum_{i=1}^m c_i = n - 1$ and $\sum_{i=1}^m c_i u_i = \vec{0}$. Now let $c'_i = \frac{c_i}{1 - \frac{1}{n^2}} = \frac{n^2 c_i}{n^2 - 1}$ for $i = 1, \dots, m$ and $c'_0 = \frac{n}{n+1}$. We then have

$$\sum_{i=1}^m c'_i v_i \otimes v_i + c'_0 (-e_n) \otimes (-e_n) = \frac{n-1}{n^2-1} e_n \otimes e_n + I_{n-1} + \frac{n}{n+1} e_n \otimes e_n = I_n.$$

Also,

$$\sum_{i=1}^m c'_i v_i - c'_0 e_n = \left(\frac{n-1}{n} \frac{n^2}{n^2-1} - \frac{n}{n+1}\right) e_n = \vec{0}.$$

□

The points $\{u_j\}_{j=1}^{2(n-1)} = \{\pm e_i\}_{i=1}^{n-1}$ with $c_j = \frac{1}{2}$ satisfy the assumption of Proposition 3.2. We set

$$A := \left\{ \left(\pm \sqrt{1 - \frac{1}{n^2}} e_i, \frac{1}{n} \right) \right\}_{i=1}^{n-1} \cup \{(\vec{0}, -1)\}$$

and

$$Q := \{x \in \mathbb{R}^n, \forall u \in A \langle x, u \rangle \leq 1\}.$$

The set A is the collection of contact points of Q . By Proposition 3.2 and Theorem 2.1, Q is in John's position.

Let $B_\infty^{n-1} := \{y \in \mathbb{R}^{n-1}, \forall i = 1, 2, \dots, n-1 \mid |\langle y, e_i \rangle| \leq 1\}$ be the unit cube in \mathbb{R}^{n-1} .

$$\begin{aligned} Q &= \{x \in \mathbb{R}^n, \forall u \in A \langle x, u \rangle \leq 1\} \\ &= \{(y, t) \in \mathbb{R}^n, y \in \frac{n-t}{\sqrt{n^2-1}} B_\infty^{n-1} \text{ and } t \in [-1, n]\}. \end{aligned}$$

Q is in John's position and it is a cone with base $\frac{n+1}{\sqrt{n^2-1}} B_\infty^{n-1}$ and height $n+1$. Thus, Q_t is $\frac{n-t}{\sqrt{n^2-1}} B_\infty^{n-1}$ for $t \in [-1, n]$. Since the radial function of B_∞^{n-1} is $\rho_{B_\infty^{n-1}}(\theta) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}}$. We have

$$(3) \quad \rho_Q(\theta, t) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}} \frac{n-t}{\sqrt{n^2-1}}.$$

3.2. Proof of Theorem 3.1(1). We define

$$(4) \quad L := \{(y, t) \in \mathbb{R}^n, y \in (2 + \frac{t}{n})B_2^{n-1} \text{ and } t \in [-1, n]\}.$$

In particular, L_t is equal to $(2 + \frac{t}{n})B_2^{n-1}$ and the radial function is $\rho_L(\theta, t) = 2 + \frac{t}{n}$.

Fix $R_0 = n - \frac{C_0}{2}\sqrt{\log(n)n}$ for some $C_0 > 0$ that we will determine later. Then, we have $\rho_L(\theta, R_0) = 3 - \frac{C_0}{2}\sqrt{\frac{\log(n)}{n}}$. By (3),

$$\rho_Q(\theta, R_0) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}} \frac{C_0}{2} \frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}}.$$

We split S^{n-2} into two components by defining

$$O_1 := \{\theta \in S^{n-2}, \rho_Q(\theta, R_0) \leq \rho_L(\theta, R_0)\}.$$

For a sufficiently large n , we have

$$\begin{aligned} O_1 &= \{\theta \in S^{n-2}, \frac{1}{(3 - \frac{C_0}{2}\sqrt{\frac{\log(n)}{n}})} \frac{C_0}{2} \frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq \max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}\} \\ &\subset \{\theta \in S^{n-2}, \frac{C_0}{6} \frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq \max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}\} \\ &\subset \bigcup_{i=1}^{n-1} \{\theta \in S^{n-2}, \frac{C_0}{6} \frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq |\langle \theta, e_i \rangle|\}. \end{aligned}$$

Due to Proposition 2.2, the measure of O_1 can be bounded:

$$\begin{aligned} \sigma_{n-2}(O_1) &\leq 4n \exp\left(-\frac{1}{36}C_3C_0^2 \frac{n^2}{n^2-1} \log(n)\right) \\ &\leq 4 \exp\left(\left(1 - \frac{1}{36}C_3C_0^2\right) \log(n)\right). \end{aligned}$$

By setting $C_0 := \sqrt{\frac{72}{C_3}}$, for a sufficiently large n , we have

$$(5) \quad \sigma_{n-2}(O_1) \leq 4 \exp(-\log(n)) \leq \frac{1}{2}.$$

Moreover, $\rho_L(\theta, t)$ is increasing with respect to $t \in [-1, n]$, while $\rho_Q(\theta, t)$ is decreasing with respect to $t \in [-1, n]$. We may conclude that,

$$(6) \quad \forall \theta \in O_1^c, \rho_Q(\theta, t) \geq \rho_L(\theta, t) \text{ for } t \in [-1, R_0].$$

We define K to be the intersection of Q and L , $K = Q \cap L$. Then, we have $K_t = Q_t \cap L_t$ and thus $\rho_K(\theta, t) = \min\{\rho_Q(\theta, t), \rho_L(\theta, t)\}$. By (2), it is sufficient to prove

$$(7) \quad \int_{S^{n-1}} \int_{-1}^n \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \geq 0,$$

with $R = n - C_0 \sqrt{\log(n)n}$. For the inner integral in (7):

$$\begin{aligned} & \int_{-1}^n \rho_K(\theta, t)^{n-1} (t - R) dt \\ & \leq \int_{-1}^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt \\ & = - \int_{-1}^R \rho_K(\theta, t)^{n-1} (R - t) dt + \int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt. \end{aligned}$$

For the first component, with $\rho_K(\theta, t) \leq \rho_L(\theta, t) = 2 + \frac{t}{n}$, we have

$$\int_{-1}^R \rho_K(\theta, t)^{n-1} (R - t) dt \leq \int_{-1}^R \left(2 + \frac{t}{n}\right)^{n-1} (R - t) dt.$$

The integral on the right side is computable via integration by parts:

$$\begin{aligned} & \int_{-1}^R \left(2 + \frac{t}{n}\right)^{n-1} (R - t) dt \\ & = \left(2 + \frac{t}{n}\right)^n (R - t) \Big|_{-1}^R + \int_{-1}^R \left(2 + \frac{t}{n}\right)^n dt \\ & = - \left(2 - \frac{1}{n}\right)^n (R + 1) + \frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1} - \frac{n}{n+1} \left(2 - \frac{1}{n}\right)^{n+1} \\ & \leq \frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1}. \end{aligned}$$

Thus,

$$(8) \quad \int_{-1}^R \rho_K(\theta, t)^{n-1} (R - t) dt \leq \frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1}.$$

For $\theta \in O_1^c$, due to (6) we have $\rho_K(\theta, t) = \rho_L(\theta, t) = \left(2 + \frac{t}{n}\right)$ for $t \in [-1, n]$. Thus, we have the equality when $\theta \in O_1^c$:

$$\int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt = \int_R^{R_0} \rho_L(\theta, t)^{n-1} (t - R) dt.$$

Again, the integral on the right side is computable:

$$\begin{aligned}
& \int_R^{R_0} \left(2 + \frac{t}{n}\right)^{n-1} (t - R) dt \\
&= \left(2 + \frac{t}{n}\right)^n (t - R) \Big|_R^{R_0} - \int_R^{R_0} \left(2 + \frac{t}{n}\right)^n dt \\
&= \left(2 + \frac{R_0}{n}\right)^n (R_0 - R) - \frac{n}{n+1} \left(2 + \frac{R_0}{n}\right)^{n+1} + \frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1}.
\end{aligned}$$

Observe that, for a sufficiently large n , we have

$$(R_0 - R) = \frac{C_1}{2} \sqrt{\log(n)n} > 4 > 2 \frac{n}{n+1} \left(2 + \frac{R_0}{n}\right).$$

Hence, the previous equality can be bounded:

$$\begin{aligned}
& \left(2 + \frac{R_0}{n}\right)^n (R_0 - R) - \frac{n}{n+1} \left(2 + \frac{R_0}{n}\right)^{n+1} + \frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1} \\
& \geq \frac{1}{2} \left(2 + \frac{R_0}{n}\right)^n (R_0 - R).
\end{aligned}$$

We conclude that, for any $\theta \in O_1^c$,

$$(9) \quad \int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt \geq \frac{1}{2} \left(2 + \frac{R_0}{n}\right)^n (R_0 - R).$$

Now we can derive the main inequality (7). First, we split the integral:

$$\begin{aligned}
& \int_{S^{n-1}} \int_{-1}^n \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\
&= \int_{S^{n-1}} \int_{-1}^R \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\
& \quad + \int_{S^{n-1}} \int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\
& \quad + \int_{S^{n-1}} \int_{R_0}^n \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta).
\end{aligned}$$

By (8), the first summand satisfies

$$\int_{S^{n-1}} \int_{-1}^R \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \geq -\frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1}.$$

According to (9) and (5), the second summand satisfies

$$\begin{aligned}
& \int_{S^{n-1}} \int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\
& \geq \int_{O_1^c} \int_R^{R_0} \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\
& \geq \frac{1}{4} \left(2 + \frac{R_0}{n}\right)^n (R_0 - R).
\end{aligned}$$

Noticing that the third summand is non-negative, we conclude that

$$\begin{aligned} & \int_{S^{n-1}} \int_{-1}^n \rho_K(\theta, t)^{n-1} (t - R) dt d\sigma_{n-2}(\theta) \\ & \geq -\frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1} + \frac{1}{4} \left(2 + \frac{R_0}{n}\right)^n (R_0 - R). \end{aligned}$$

With $\frac{1}{4}(R_0 - R) > 2 > \frac{n}{n+1} \left(2 + \frac{R}{n}\right)$ and $\left(2 + \frac{R_0}{n}\right)^n > \left(2 + \frac{R}{n}\right)^n$, we get

$$-\frac{n}{n+1} \left(2 + \frac{R}{n}\right)^{n+1} + \frac{1}{4} \left(2 + \frac{R_0}{n}\right)^n (R_0 - R) > 0$$

for a sufficiently large n . Hence,

$$\int_{S^{n-1}} \int_{-1}^n \rho_K(\theta, t) (t - R) dt d\sigma_{n-2}(\theta) > 0.$$

We conclude from (2) that

$$|x_K| > R = n - C_0 \sqrt{\log(n)n} = (1 - C_0 \sqrt{\frac{\log(n)}{n}})n.$$

3.3. Proof of Theorem 3.1 (2). To construct P in Theorem 3.1 (2) we define a cylinder L_2 , which is the intersection of $O(n^2)$ number of half spaces and set $P := Q \cap L_2$, where Q is the same as above.

Let $\{\epsilon_n\}$ be a decreasing sequence. Later we will specify ϵ_n , but for now we assume that

$$(10) \quad \frac{10}{n} < \epsilon_n < 1, \text{ and}$$

$$(11) \quad \lim_{n \rightarrow +\infty} \epsilon_n = 0.$$

Let

$$A' := \{\pm(1 - \epsilon_n)e_i \pm \sqrt{1 - (1 - \epsilon_n)^2}e_j\}_{i,j < n, i \neq j},$$

and

$$L_2 := \{(y, t) \in \mathbb{R}^n, \langle y, u \rangle \leq 1 \ \forall u \in A' \text{ and } t \in [-1, n]\}.$$

We have $|A'| = 4n(n-1)$ and L_2 is a cylinder with

$$L_{2,t} = \{y \in \mathbb{R}^{n-1}, \langle y, u \rangle \leq 1 \ \forall u \in A'\}$$

for $t \in [-1, n]$. Let $P = Q \cap L_2$. Since $B_2^n \subset L_2$ and Q is in John's position, P is in John's position. Following the same approach from the proof of Theorem 3.1 (1), we want to show

$$(12) \quad \int_{S^{n-2}} \int_{-1}^n \rho_P(\theta, t)^{n-1} \left(t - \frac{1}{5}\epsilon_n n\right) dt d\sigma_{n-2}(\theta) > 0.$$

Then, we can conclude $|x_P| > \frac{1}{5}\epsilon_n n$.

For convenience, let $Q' := Q_{\epsilon_n n}$ and $L' := L_{2, \epsilon_n n}$. Also, let $\rho_{Q'}(\cdot) := \rho_Q(\cdot, \epsilon_n n)$ and $\rho_{L'}(\cdot) := \rho_{L_2}(\cdot, \epsilon_n n)$. We will show that for the majority of $\theta \in S^{n-2}$, $\rho_P(\theta, t) = \rho_{L_2}(\theta, t)$ for $t \in [-1, \epsilon_n n]$. In the case that $\rho_P(\theta, t) \neq \rho_{L_2}(\theta, t)$ for some t in $[-1, \epsilon_n n]$, $\rho_P(\theta, t)$ will be nicely bounded.

Proposition 3.3. *With the notation above, let*

$$O_2 := \{\theta \in S^{n-2}, \rho_{Q'}(\theta) \leq \rho_{L'}(\theta)\}.$$

For a sufficiently large n , we have

- (1) $\forall \theta \in O_2, \rho_{Q'}(\theta) \leq 4\sqrt{\epsilon_n n}$.
- (2) $\sigma_{n-2}(O_2) \leq 4n \exp(-\frac{C_6}{\epsilon_n})$, where $C_6 > 0$ is a universal constant.

Proof. Let $y \in \partial Q' \cap L'$. Then, there exists i such that $|y_i| = (1 - \epsilon_n) \frac{n}{\sqrt{n^2 - 1}} = \rho_{Q'}(\frac{y}{|y|})$. Following the conditions from the definition of L' , we have, for $j \neq i$,

$$\begin{aligned} (1 - \epsilon_n)|y_i| + \sqrt{1 - (1 - \epsilon_n)^2}|y_j| &\leq 1 \\ \Rightarrow \sqrt{1 - (1 - \epsilon_n)^2}|y_j| &\leq 1 - (1 - \epsilon_n)^2 \\ \Rightarrow |y_j| &\leq \sqrt{1 - (1 - \epsilon_n)^2}, \end{aligned}$$

where for the second inequality we use $\frac{n}{\sqrt{n^2 - 1}} \geq 1$.

From the previous argument, $y \in \partial Q' \cap L'$ implies that

$$(13) \quad |y| \leq \sqrt{(n-2)(1 - (1 - \epsilon_n)^2) + (1 - \epsilon_n)^2 \frac{n^2}{n^2 - 1}}.$$

By (10) and (11), we have $0 < (1 - (1 - \epsilon_n)^2) = 2\epsilon_n - \epsilon_n^2 \leq 2\epsilon_n$ and $n\epsilon_n > 1$. Hence, (13) becomes

$$\begin{aligned} |y| &\leq \sqrt{(n-2)(1 - (1 - \epsilon_n)^2) + (1 - \epsilon_n)^2 \frac{n^2}{n^2 - 1}} \\ &\leq \sqrt{2\epsilon_n n + 2} \\ &\leq 2\sqrt{\epsilon_n n}, \end{aligned}$$

which proves Claim (1) in Proposition 3.3.

For $\theta \in O_2$, $\rho_Q(\theta)\theta \in \partial Q' \cap L'$. There exists i such that $|(\rho_{Q'}(\theta)\theta)_i| = (1 - \epsilon_n) \frac{n}{\sqrt{n^2 - 1}}$. By (11), $(1 - \epsilon_n) \frac{n}{\sqrt{n^2 - 1}} > \frac{1}{2}$ for large n . Together with $\rho_{Q'}(\theta) \leq 2\sqrt{\epsilon_n n}$,

$$(14) \quad |\theta_i| = \frac{(1 - \epsilon_n) \frac{n}{\sqrt{n^2 - 1}}}{\rho_{Q'}(\theta)} \geq \frac{1}{2\rho_{Q'}(\theta)} \geq \frac{1}{4\sqrt{\epsilon_n n}}.$$

Thus, inequality (14) leads to the following inclusion:

$$O_2 \subset \cup_{i=1}^{n-1} \{\theta \in S^{n-2}, |\theta_i| \geq \frac{1}{4\sqrt{\epsilon_n n}}\}.$$

By Proposition 2.2,

$$\sigma_{n-2}(\{\theta, |\theta_i| \geq \frac{1}{4\sqrt{\epsilon_n n}}\}) \leq 4 \exp(-\frac{C_3}{16\epsilon_n} \frac{n-1}{n}) \leq 4 \exp(-\frac{C_6}{\epsilon_n}).$$

Therefore, using the union bound, we conclude that

$$\sigma_{n-2}(O_2) \leq 4n \exp(-\frac{C_6}{\epsilon_n}).$$

□

Proposition 3.4. *With the notation above, there exists a constant $C_7 > 0$ such that if the sequence $\{\epsilon_n\}$ satisfies $\frac{C_7}{\log(n)} > \epsilon_n$ for a large sufficiently n , then*

$$\sigma_{n-2}(\{\theta, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq 4 \exp(-\frac{C_8}{\epsilon_n}),$$

where $C_8 > 0$ is a universal constant.

Proof. Let $\|\cdot\|$ be the norm on \mathbb{R}^{n-1} such that L' is the unit ball that corresponds to the norm $\|\cdot\|$. More specifically, for $y \in \mathbb{R}^{n-1}$,

$$\|y\| = \max_{1 \leq i, j < n, i \neq j} \{(1 - \epsilon_n)|y_i| + \sqrt{1 - (1 - \epsilon_n)^2}|y_j|\}.$$

Let $g = (g_1, g_2, \dots, g_{n-1})$ be the standard Gaussian random vector in \mathbb{R}^{n-1} . Then,

$$\begin{aligned} \mathbb{E}\|g\| &= \mathbb{E} \max_{1 \leq i, j < n, i \neq j} \{(1 - \epsilon_n)|g_i| + \sqrt{1 - (1 - \epsilon_n)^2}|g_j|\} \\ &\leq 2\mathbb{E} \max_{i=1, \dots, n-1} |g_i| \leq c' \sqrt{\log(n)}, \end{aligned}$$

where $c' > 0$ is a universal constant and the last inequality is a classical result for the extreme value of independent Gaussian random variables.

Using the standard polar integration, we obtain the following inequality,

$$\int_{S^{n-2}} \|\theta\| d\sigma_{n-2}(\theta) \leq \frac{c''}{\sqrt{n}} \mathbb{E}\|g\|,$$

where $c'' > 0$ is a universal constant. Thus, $\mathbb{E}_{\sigma_{n-2}} \|\theta\| \leq c' c'' \sqrt{\frac{\log(n)}{n}}$. Moreover, $\sup_{\theta \in S^{n-2}} \|\theta\| \leq 1$ due to the fact that $B_2^{n-1} \subset L'$. Therefore, the function $\theta \rightarrow \|\theta\|$ is 1-Lipschitz on S^{n-2} . We set $C_7 > 0$ to be small enough so that $\frac{1}{2} \frac{1}{5\sqrt{\epsilon_n n}} > c' c'' \sqrt{\frac{\log(n)}{n}}$. Since $\rho_{L'}(\theta) = \frac{1}{\|\theta\|}$, we have the equality

$$\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\} = \{\theta \in S^{n-2}, \|\theta\| \geq \frac{1}{5\sqrt{\epsilon_n n}}\}.$$

Furthermore, the inequality $\mathbb{E}_{\sigma_{n-2}} \|\theta\| \leq \frac{1}{2} \frac{1}{5\sqrt{\epsilon_n n}}$ implies

$$\{\theta \in S^{n-2}, \|\theta\| \geq \frac{1}{5\sqrt{\epsilon_n n}}\} \subset \{\theta \in S^{n-2}, ||\|\theta\| - \mathbb{E}\|\theta\|| > \frac{1}{10\sqrt{\epsilon_n n}}\}.$$

Together with Theorem 2.3, we may conclude that

$$\begin{aligned} & \sigma_{n-2}(\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \\ & \leq \sigma_{n-2}(\{\theta \in S^{n-2}, ||\|\theta\| - \mathbb{E}\|\theta\|| > \frac{1}{10\sqrt{\epsilon_n n}}\}) \\ & \leq 4 \exp\left(-\frac{C_8}{\epsilon_n}\right), \end{aligned}$$

where we use Theorem 2.3 in the last inequality. \square

Now we are able to prove Theorem 3.1 (2).

Proof of Theorem 3.1 (2). We want to choose ϵ_n so that

$$(15) \quad \sigma_{n-2}(O_2) < \frac{1}{4},$$

and

$$(16) \quad \sigma_{n-2}(\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq \frac{1}{4},$$

for a large n .

According to Proposition 3.3, the first condition can be achieved if $\epsilon_n < \frac{c}{\log(n)}$ for some $c > 0$ when n is large.

Moreover, we also want to choose $\epsilon_n < \frac{c'}{\log(n)}$ so that we can apply Proposition 3.4 to get $\sigma_{n-2}(\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq \frac{1}{4}$.

Therefore, we can set $\epsilon_n = \frac{c''}{\log(n)}$ for some $c'' > 0$ so that (15) and (16) hold.

Recall that from (12) our goal is to show that

$$\int_{S^{n-2}} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) > 0.$$

Since $P = Q \cap L_2$,

$$\rho(\theta, t) = \min\{\rho_Q(\theta, t), \rho_{L_2}(\theta, t)\} = \min\{\rho_Q(\theta, t), \rho_{L'}(\theta)\}.$$

We handle the inner integral differently for $\theta \in O_2$ and $\theta \notin O_2$.

- In the case that $\theta \notin O_2$:

First, we have $\rho_Q(\theta, \epsilon_n n) \geq \rho_{L_2}(\theta, \epsilon_n n)$. Thus, $\rho_P(\theta, t) = \rho_{L'}(\theta)$ for $t \in [-1, \epsilon_n n]$. This is because $\rho_{L'}(\theta)$ is a constant and $\rho_P(\theta, t)$ is decreasing with respect to t . Thus,

$$\begin{aligned} & \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & \geq \int_{-1}^{\epsilon_n n} \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & = \int_{-1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt. \end{aligned}$$

We split the integral to two parts:

$$\begin{aligned} & \int_{-1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & = \int_{-1}^{2\frac{1}{5}\epsilon_n n + 1} \rho_{L'}(\theta)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & \quad + \int_{2\frac{1}{5}\epsilon_n n + 1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt. \end{aligned}$$

Due to the symmetry of the integrand with respect to $t = \frac{1}{5}\epsilon_n n$, the first summand is 0. For the second summand, we have

$$\begin{aligned} & \int_{2\frac{1}{5}\epsilon_n n + 1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & \geq (\epsilon_n n - \frac{2}{5}\epsilon_n n - 1) \rho_{L'}(\theta)^{n-1} (\frac{1}{5}\epsilon_n n + 1) \\ & \geq \frac{(\epsilon_n n)^2}{10} \rho_{L'}(\theta)^{n-1}, \end{aligned}$$

where in the second to last inequality we used that $\frac{2}{5}\epsilon_n n + 1 \leq \frac{1}{2}\epsilon_n n$ by (10). We conclude that

$$(17) \quad \forall \theta \in O_2^c, \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \geq \frac{(\epsilon_n n)^2}{10} \rho_{L'}(\theta)^{n-1}.$$

- In the case that $\theta \in O_2$:

From Proposition 3.3, we know that $\rho_Q(\theta, \epsilon_n n) \leq 2\sqrt{\epsilon_n n}$. Therefore, since $\rho_Q(\theta, t)$ is linear on $[-1, n]$ and $\rho_Q(\theta, n) = 0$, we see that for any $t \in [-1, n]$,

$$\rho_Q(\theta, t) \leq \frac{n+1}{n-\epsilon_n n} 2\sqrt{\epsilon_n n} \leq 4\sqrt{\epsilon_n n},$$

for a sufficiently large n . We have

$$\int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \geq \int_{-1}^{\frac{1}{5}\epsilon_n n} \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt,$$

because the integrand is positive for $t > \frac{1}{5}\epsilon_n n$. Then, using the estimate of $\rho_Q(\theta, t) \leq 4\sqrt{\epsilon_n n}$,

$$\begin{aligned} & \int_{-1}^{\frac{1}{5}\epsilon_n n} \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \\ & \geq -(\frac{1}{5}\epsilon_n n + 1)(4\sqrt{\epsilon_n n})^{n-1}(1 + \frac{1}{5}\epsilon_n n) \\ & \geq -\frac{4}{25}(\epsilon_n n)^2(4\sqrt{\epsilon_n n})^{n-1}, \end{aligned}$$

where in the last inequality we used $\frac{1}{5}\epsilon_n n + 1 \leq \frac{2}{5}\epsilon_n n$, which is valid for a large n . Therefore, we have

$$(18) \quad \forall \theta \in O_2, \quad \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt \geq -\frac{4}{25}(\epsilon_n n)^2(4\sqrt{\epsilon_n n})^{n-1}.$$

Now we are able to derive the main inequality.

$$\begin{aligned} & \int_{S^{n-2}} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) \\ (19) \quad & = \int_{O_2} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) \\ & + \int_{O_2^c} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta). \end{aligned}$$

Applying (17), the second summand satisfies

$$\int_{O_2^c} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) \geq (\epsilon_n n)^2 \int_{O_2^c} \frac{1}{10} \rho_{L'}(\theta)^{n-1} d\sigma_{n-2}(\theta).$$

Let $U := \{\theta, \rho_{L'}(\theta) \geq 5\sqrt{\epsilon_n n}\}$. From (15) and (16) we know that $\sigma_{n-2}(U \cap O_2^c) \geq \frac{1}{2}$ for a large n . Since the integrand is positive,

$$\int_{O_2^c} \frac{1}{10} \rho_{L'}(\theta)^{n-1} d\sigma_{n-2}(\theta) \geq \int_{U \cap O_2^c} \frac{1}{10} \rho_{L'}(\theta)^{n-1} d\sigma_{n-2}(\theta).$$

Thus,

$$\int_{O_2^c} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5} \epsilon_n n) dt d\sigma_{n-2}(\theta) \geq (\epsilon_n n)^2 \int_{U \cap O_2^c} \frac{1}{10} \rho_L(\theta)^{n-1} d\sigma_{n-2}(\theta).$$

For the first summand of (19), we apply (18) and (15) to get

$$\begin{aligned} & \int_{O_2} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5} \epsilon_n n) dt d\sigma_{n-2}(\theta) \\ & \geq -(\epsilon_n n)^2 \sigma_{n-2}(O_2) \frac{4}{25} (4\sqrt{\epsilon_n n})^{n-1}. \end{aligned}$$

Combining the inequalities for the two summands together we have

$$\begin{aligned} & \int_{S^{n-2}} \int_{-1}^n \rho_P(\theta, t)^{n-1} (t - \frac{1}{5} \epsilon_n n) dt d\sigma_{n-2}(\theta) \\ & \geq (\epsilon_n n)^2 \left[\frac{1}{20} (5\sqrt{\epsilon_n n})^{n-1} - \frac{1}{25} (4\sqrt{\epsilon_n n})^{n-1} \right] \\ & \geq 0. \end{aligned}$$

Therefore, the center of mass is at least $C_1 \frac{n}{\log(n)}$ away from 0, where $C_1 := \frac{c''}{5}$.

□

4. THE RELATION BETWEEN THE CONJECTURES

Let $K \subset \mathbb{R}^n$ be a convex body in John's position and X be a random vector uniformly distributed in K . Let M_K denote the median of $|X|$, which is the unique value satisfying

$$\mathbb{P}(|X| \leq M_K) = \frac{1}{2}.$$

Lemma 4.1. *Let $K \subset \mathbb{R}^n$ be a convex body. Let X be a random vector uniformly distributed in K . Let M_K denote the median of $|X|$. Then, we have*

$$\frac{M_K}{\sqrt{2}} \leq (\mathbb{E}|X|^2)^{1/2} \leq C_9 M_K,$$

where $C_9 > 0$ is a universal constant.

Proof. The first inequality is standard:

$$\mathbb{E}|X|^2 \geq \mathbb{E}(|X|^2 1_{|X| \geq M_K}) \geq \frac{1}{2} M_K^2$$

Thus, the first inequality can be obtained by taking square root on both sides.

To prove the second one, let R be the number such that $\mathbb{P}(|X| \leq R) = \frac{2}{3}$. We can apply Theorem 2.4 with $U = RB_2^n$ and $\delta = \frac{2}{3}$ to get

$$\mathbb{P}(|X| > tR) \leq \frac{\sqrt{2}}{3} 2^{-t/2} \text{ for } t > 1.$$

A simple integration shows that

$$\mathbb{E}|X|^2 \leq cR^2,$$

for a universal constant $c > 0$.

Now we apply Theorem 2.6 with $b = \frac{2}{3}$ and $U = RB_2^n$ to obtain

$$\mathbb{P}(|X| \leq M_K B_2^N) \leq C_b \frac{M_K}{R} \mathbb{P}(|X| \leq R),$$

which implies that $M_K \geq \frac{3}{C_{2/3}} R$.

□

We could also relate $\mathbb{E}|X|^2$ and the center of mass of K , x_K , when K is in John's position.

Lemma 4.2. *There exists $C_{10}, C_{11} > 0$ such that, for any convex body $K \subset \mathbb{R}^n$ in John's position, we have*

$$|x_K|^2 \leq \mathbb{E}|X|^2 \leq C_{10}|x_K|^2 + C_{11}n,$$

where X is a random vector uniformly distributed in K .

This result was proved by M. Fradelizi, G. Paouris and C. Schütt in [3].

Here we present a different proof.

Proof. Since K is in John's position, there exists $\{u_i\}_{i=1}^m \subset S^{n-1}$ and $\{c_i\}_{i=1}^m$ with $c_i > 0$ such that $\sum_{i=1}^m c_i u_i = 0$ and $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. In particular,

$$\mathbb{E}|X|^2 = \mathbb{E} \sum_{i=1}^m c_i (\langle X, u_i \rangle)^2.$$

Also, $|x_K|^2 = \sum_{i=1}^m c_i (\langle x_K, u_i \rangle)^2$.

Given that u_i is a contact point of K , we have $\langle x, u_i \rangle \leq 1$ for all $x \in K$. As a consequence, with $\langle x_K, u_i \rangle = \mathbb{E}\langle X, u_i \rangle$ we have

$$0 \leq \mathbb{E}|\langle X, u_i \rangle| - |\langle x_K, u_i \rangle| \leq 2.$$

Here, the first inequality follows from Jensen's inequality while the second one relies on an elementary observation that for any random variable $Y \leq 1$, $\mathbb{E}|Y| = 2\mathbb{E}\max\{Y, 0\} - \mathbb{E}Y \leq 2 + |\mathbb{E}Y|$. Thus,

$$|\langle x_K, u_i \rangle|^2 \leq (\mathbb{E}|\langle X, u_i \rangle|)^2 \leq 3(|\langle x_K, u_i \rangle|^2 + 2).$$

According to Theorem 2.5, we have

$$\mathbb{E}|\langle X, u_i \rangle| \leq (\mathbb{E}|\langle X, u_i \rangle|^2)^{1/2} \leq 2C_5\mathbb{E}|\langle X, u_i \rangle|.$$

Therefore, we can conclude that

$$|x_K|^2 \leq \mathbb{E}|X|^2 \leq 12C_5(|x_K|^2 + 2 \sum c_i) \leq C|x_K|^2 + C'n,$$

where the last inequality uses the fact that $\sum_{i=1}^m c_i = n$. □

Corollary 4.3. *Conjecture 1.2 and Conjecture 1.1 are equivalent.*

Proof. Let $K \subset \mathbb{R}^n$ be a convex body. Since the result is invariant under affine transformations, we may assume that K is in John's position. Let X be a random vector uniformly distributed in K and M_K be the median of the random variable $|X|$.

Suppose Conjecture 1.1 is true. There exists a universal constant $C > 0$ such that $|x_K| \leq C\sqrt{n}$. According to Lemma 4.1 and Lemma 4.2,

$$\begin{aligned} M_K &\leq \sqrt{2}(\mathbb{E}|X|^2)^{1/2} \\ &\leq \sqrt{2}\sqrt{C_{10}|x_K|^2 + C_{11}n} \\ &\leq \sqrt{2n}\sqrt{C_{10}C^2 + C_{11}}. \end{aligned}$$

This argument is valid for any convex body K ; therefore, Conjecture 1.2 is true.

On the other hand, assuming Conjecture 1.2 is valid, there exists a universal constant $C > 0$ such that $M_K \leq C\sqrt{n}$. Again, according to Lemma 4.1 and Lemma 4.2,

$$\begin{aligned} |x_K| &\leq (\mathbb{E}|X|^2)^{1/2} \\ &\leq C_9M_K \\ &\leq C_9C\sqrt{n}. \end{aligned}$$

Therefore, Conjecture 1.1 is true. □

The examples in Corollary 1.4 will be examples K, P , which are constructed in Theorem 3.1. For Corollary 1.4(2), the result will follow by $|x_P| \leq C_9 M_P$. Corollary 1.4(1) is a more delicate situation, and so the same argument does not apply. Observe that, for $R > 0$,

$$K \cap RB_2^n \subset K \cap \{x \in \mathbb{R}^n, \langle x, e_1 \rangle \leq R\}.$$

It is sufficient to show a stronger statement:

$$\text{vol}(K \cap \{x \in \mathbb{R}^n, \langle x, e_n \rangle \leq R\}) \leq \text{vol}(K \cap \{x \in \mathbb{R}^n, \langle x, e_1 \rangle > R\})$$

for $R = n - C'_0 \sqrt{\log(n)n}$. Adapting the notations from the proof, this is equivalent to show

$$\int_{S^{n-1}} \int_{-1}^n \rho_K(\theta, t)^{n-1} \text{sign}(t - R) dt d\sigma_{n-2} > 0.$$

The proof of this statement is almost identical to the proof of Theorem 3.1(1).

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